Applying IRS Multi-Mode Templates to Parameter Estimation

Bernard J. Kelly

with John G. Baker

NASA Goddard Space Flight Center

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Gravitational Waves from Numerical Mergers

NR black-hole merger simulations produce waveforms decomposed into (spin-weighted) spherical harmonics: $r\psi_4(t, r, \theta, \phi) = \sum_{\ell m} C_{\ell m}(t, r) \, _{-2}Y_\ell^m(\theta, \phi)$.

- We work with *strain-rate* $\dot{h} = \int \psi_4^* dt$; modal power $\dot{E}_{\ell m} \propto \dot{h}_{\ell m}^2$

- Each mode has an amplitude and complex phase: $r\dot{h}_{\ell m} = A_{\ell m}(t)e^{i\varphi_{\ell m}(t)}$.

- A handful of modes dominate energy flux; mostly $(\ell, \pm\ell)$.

- $(2, \pm 2)$ is sufficient for *detection*; other modes are important for *parameter estimation*.

Modal power for `X1_UU` (left), and `X4_00` (right).
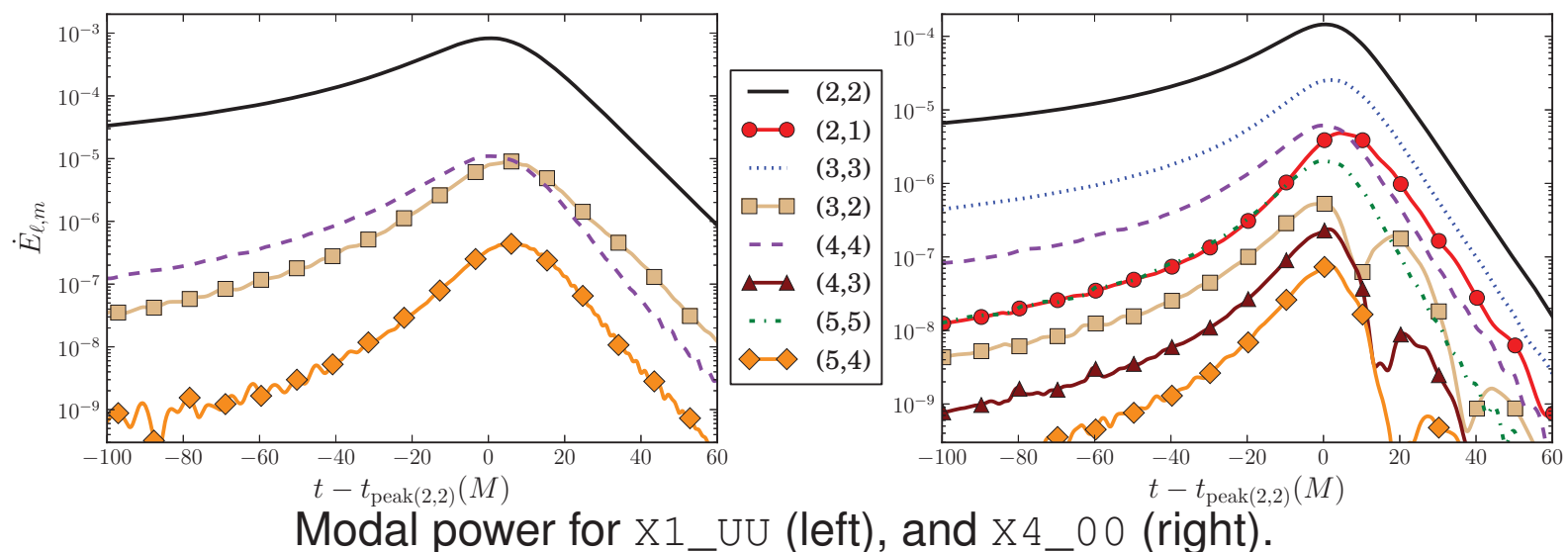
Dominant Frequency Behavior

Baker *et al.* (2008) noted that many important modes have common *rotational* frequency $\Omega_{\ell m} \equiv \omega_{\ell m}/m$.
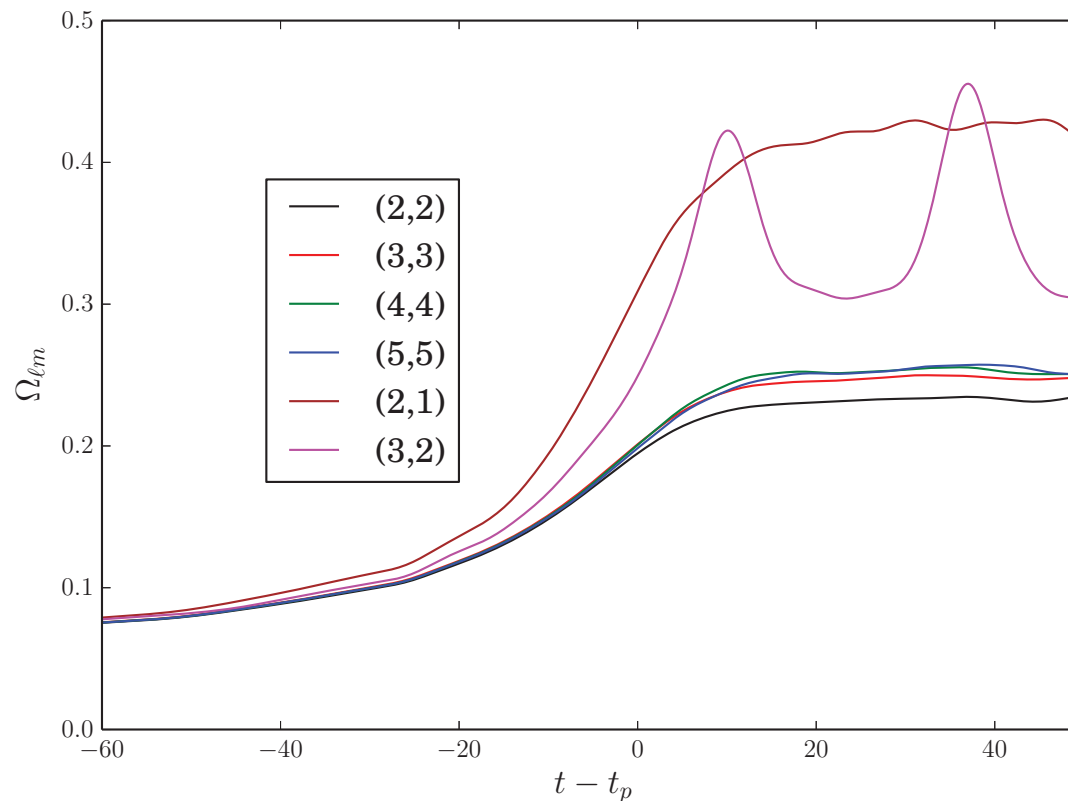
Rotational frequency for several (ℓ, m) modes for 4:1 nonspinning merger.
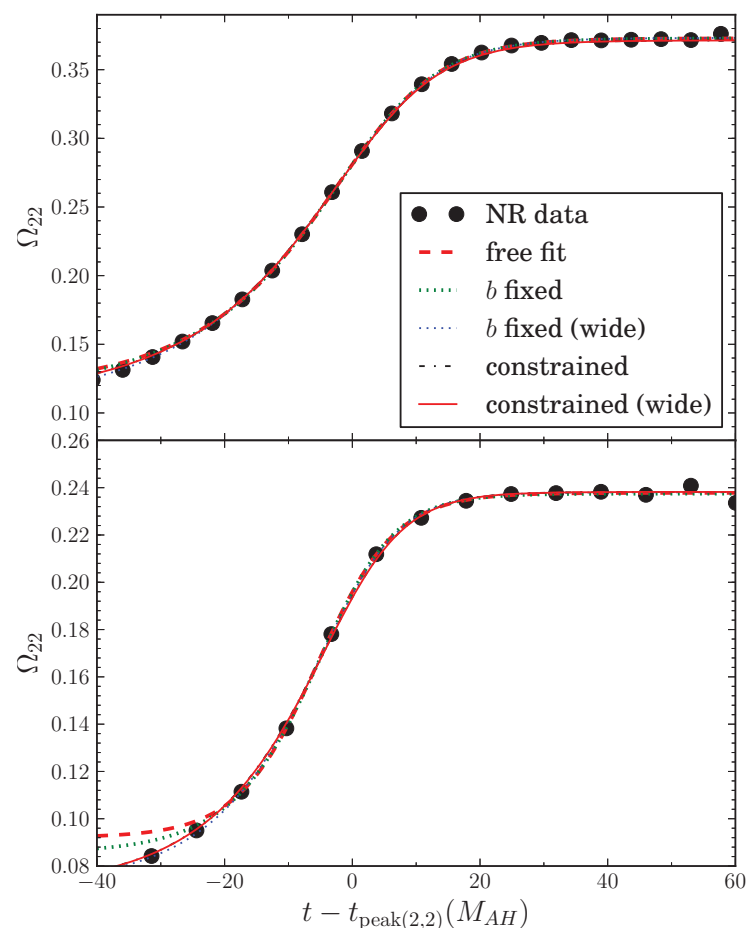
Implicit Rotating Source Picture

[Baker *et al.* (2008), Kelly *et al.* (2011)]:

- Most important WF modes had consistent *rotational phases* $\Phi_{\ell m} \equiv \varphi_{\ell m}/m$ *through merger*.

- Best matches are for $\ell = m$ modes.

- Rotational frequency model is a smoothed "step function" to fundamental QNM frequency:

$$\Omega(t) \quad \equiv \quad \dot{\Phi} = \Omega_{\mathrm{f}}(1 - f(t)),$$

$$f(t) \quad = \quad C\left[1 - \left(1 + \alpha e^{-2t/b}\right)^{-\kappa}\right].$$

- Poor for times earlier than $\sim 60M$ before merger.

Fit of $\Omega(t)$ for $(2,2)$ mode of `X1_UU` (top) and `X1_DD` (bottom).
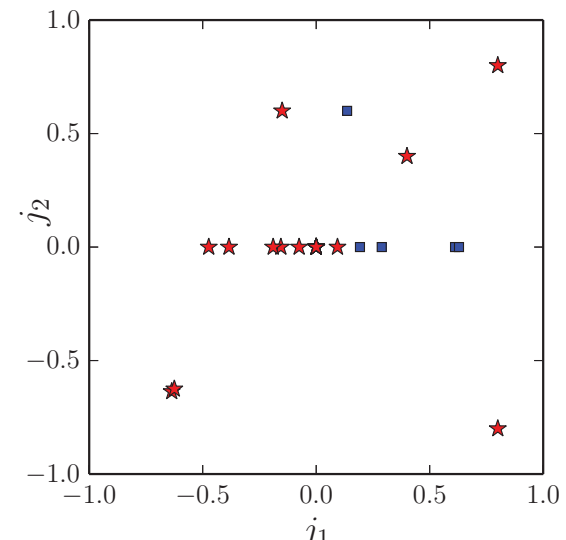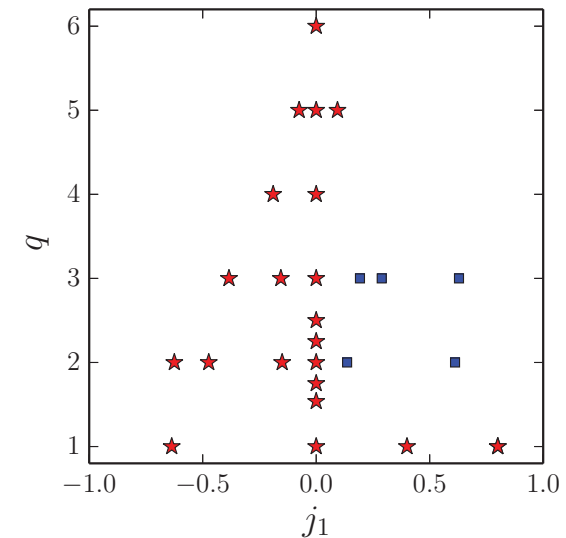
Modeling IRS *Parameters*

- Assemble broad set of aligned-spin BHB merger configurations.

- Collect free parameters $\{C, \kappa, \alpha\}$ over all configurations.

- Easier to model cuts along BH parameter directions . . .

- Symmetric mass ratio
$\eta \equiv M_1 M_2/(M_1 + M_2)^2 \le 0.25$

- "Total" spin $\tilde{\jmath} \equiv (q^2 j_1 + j_2)/(q^2 + 1)$

- Simplest fit model is product of mass-ratio and spin forms:

$$
\begin{aligned}
C(\eta, \tilde{\jmath}) &= g(\eta) \cdot h(\tilde{\jmath}) \\
g(\eta) &= g_0 + g_1(\eta_0 - \eta) + g_2(\eta_0 - \eta)^2, \\
h(\tilde{\jmath}) &= 1 + h_1 \tilde{\jmath} + h_2 \tilde{\jmath}^2.
\end{aligned}
$$

BHB configurations used.
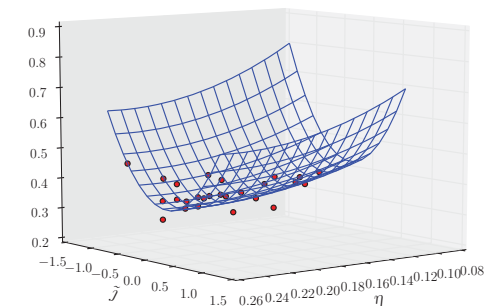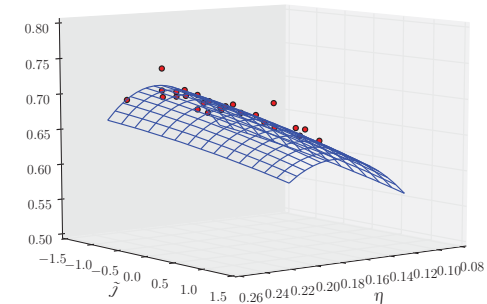
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Fit to C [top] and κ [bottom] over all configurations.

Modeling IRS *Parameters*

Remaining IRS parameters depend on the end-state Kerr hole: $\Omega_f(M_f, a_f)$ & $b(M_f, a_f)$.

- We want full initial-configuration prescription:

$$\{M_1, M_2, \vec{S}_1, \vec{S}_2\} \rightarrow \{\Omega_f, b\}$$

- Many prescriptions available, covering different ranges (nonspinning, aligned-spin, generic-spin), e.g. Lousto *et al.* (2010), Tichy & Marronetti (2008), Rezzolla *et al.* (2008), Barausse & Rezzolla (2009), Lousto & Zlochower (2014) . . .

- Use the simplest prescription consistent with aligned-spin BHBs:

$$M_f = 1 - \eta E_{\text{ISCO}} - E_2 \eta^2 - E_3 \eta^3$$

$$- \frac{\eta^2}{(1+q)^2} (E_S(j_2 + q^2 j_1) + E_\delta(1-q)(j_2 - q j_1) + E_A(j_2 + q j_1)^2) + E_D(j_2 - q j_1)^2$$

$$\frac{a_f}{M_f} \equiv j_f = \tilde{j} + \tilde{j}\eta(s_4 \tilde{j} + s_5 \eta + t_1) + \eta(2\sqrt{(3)} + t_2 \eta + t_3 \eta^2)$$

- . . . not necessarily the *best* prescription.

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- We want full initial-configuration prescription:

$$\{M_1, M_2, \vec{S}_1, \vec{S}_2\} \xrightarrow{\text{end-state model}} \{M_f, a_f\} \xrightarrow{\text{QNM theory}} \{\Omega_f, b\}$$

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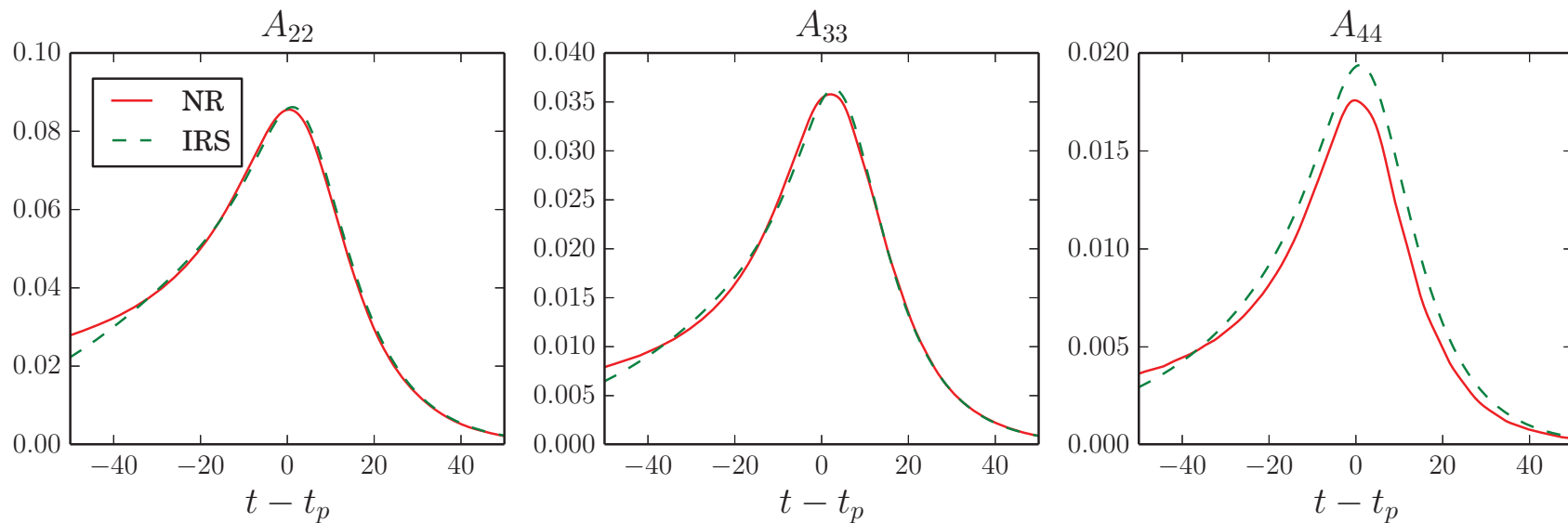- ... not necessarily the *best* prescription.

Modeling IRS Multimode Amplitudes

IRS model [Kelly *et al.* (2011)] suggested a general form for IRS amplitude functions:

$$A_{\ell m}(t) = A_0 \sqrt{\frac{\left|\dot{f}(t)\right|}{1 + a_1\left(f^2 - f^4\right) + a_2\left(f^4 - f^6\right)}}.$$

Three free parameters for each (ℓ, m) pair: A_0, a_1, a_2.

A_{22}, A_{33}, and A_{44} for 4:1 nonspinning merger.
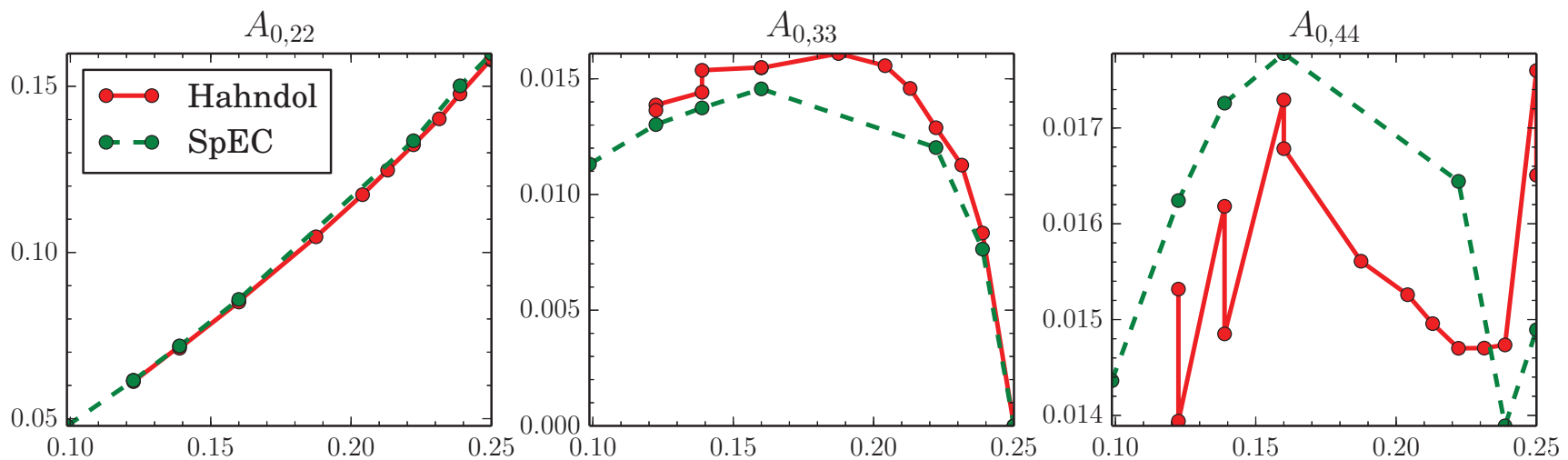
Modeling IRS Amplitude Parameters

Amplitude parameters $A_{0,lm}$ don't generally work well with quadratic forms; use leading-order post-Newtonian η-scaling as guidance:

$$rh_{22} \propto \eta \left[1 + x\left(-\frac{107}{42} + \frac{55\eta}{42} \right) + O_{3/2} \right] \qquad \textit{GOOD}$$

$$rh_{33} \propto \eta \sqrt{1 - 4\eta} \left[x^{1/2} + x^{3/2}(-4 + 2\eta) + O_2 \right] \qquad \textit{GOOD}$$

$$rh_{44} \propto \eta \left[x(1 - 3\eta) + \frac{x^2}{22}\left(-\frac{593}{5} + \frac{1273\eta}{3} - 175\eta^2 \right) + O_3 \right] \qquad \textit{BAD}$$
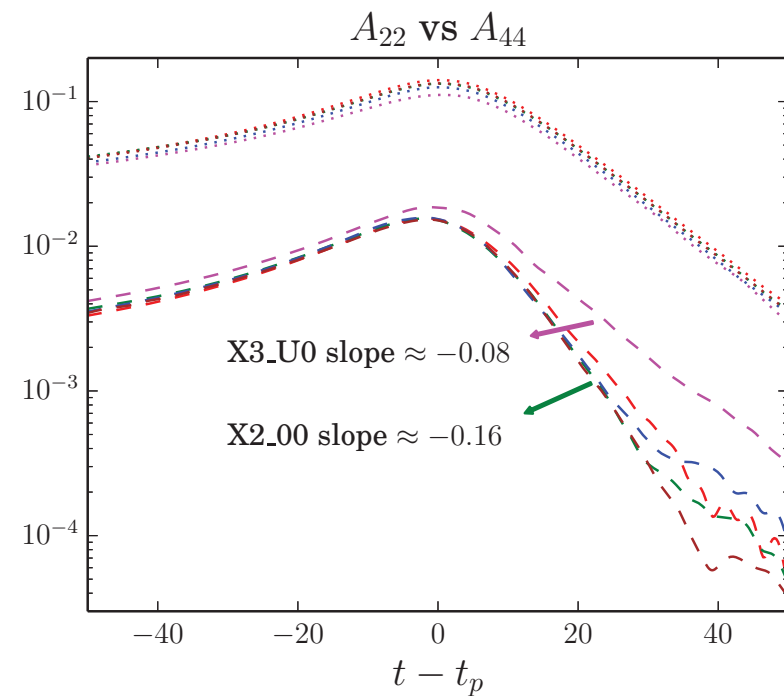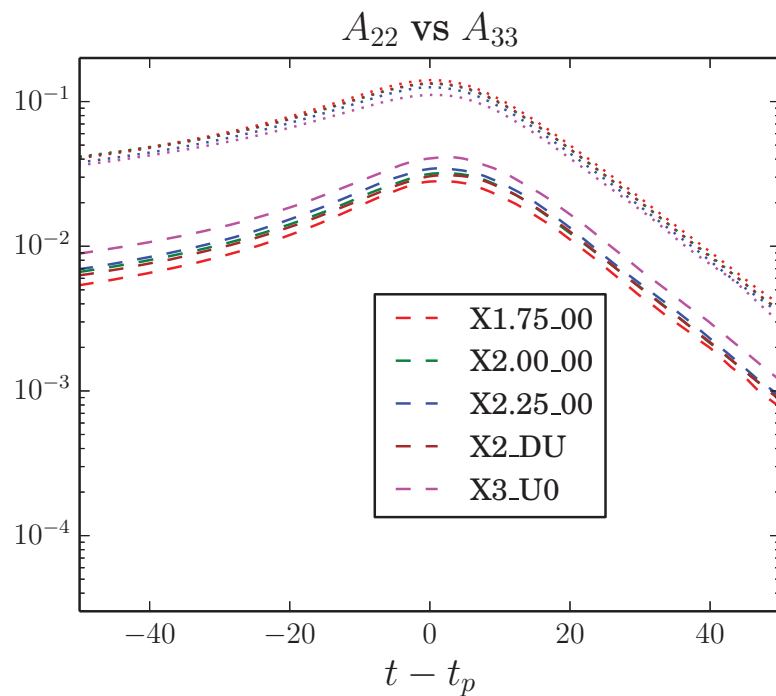

$A_{0,22}$, $A_{0,33}$, and $A_{0,44}$ parameters for nonspinning configurations.

Other problems with A_{44}

Something funny happens to A_{44} near $q = 2$ ...

... fall-off too quick for $n = 0$ QNM ($1/\tau \approx 0.08$); too slow for $n = 1$ QNM ($1/\tau \approx 0.24$) — mode-mixing?

Dealing with the $(3, 2)$ Mode

Kelly & Baker (2013) showed that "observed" $(3, 2)$ mode at merger is largely $(2, 2)$ mode, leaked through mismatch between *spherical* harmonics $_{-2}Y_{\ell'}^m$ and *spheroidal* harmonics $_{-2}\mathcal{Y}_\ell^m$.
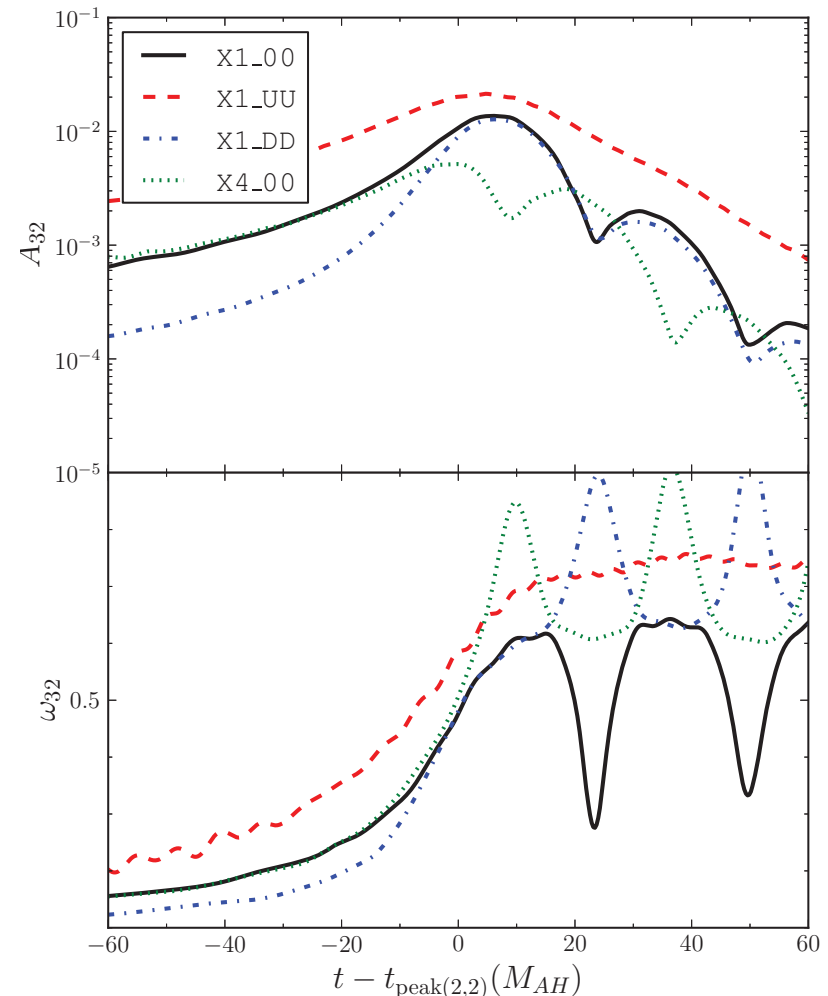
- QNM eigenfunctions are *spheroidal* harmonics

- Overlap with *spherical* harmonics is

$$s_{\ell'\ell m} = \oint d\Omega \, _{-2}\mathcal{Y}_\ell^m(a_{\mathrm{f}}\sigma_{22}; \theta, \phi) \, _{-2}Y_{\ell'}^m(\theta, \phi)^*$$

- ...leading to mixing coefficients

$$\rho_{\mathrm{basis}, \ell 2} \equiv \frac{s_{\ell'22}}{s_{2'22}}$$

- Fits observed $(3, 2)$ modes very well; $(4, 2)$ numerics too uncertain

- Can/should project *all* WFs onto spheroidal basis?

Amplitude and frequency for several runs.

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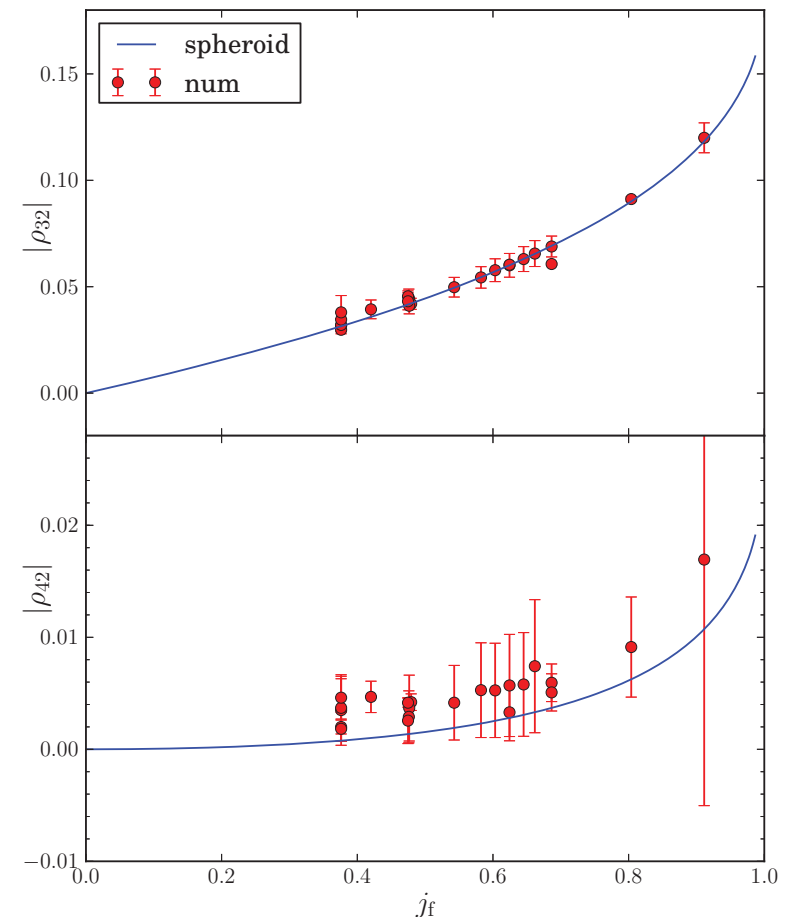
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Observed mixing for $(3, 2)$ and $(4, 2)$ modes.

Summary

- Need: higher-quality, higher-mode waveforms — NRAR? SpEC?

- Need: treatment of important $\ell \neq m$ modes (e.g. (2,1), (3,2))

- Need: better behavior of pre-merger IRS segment — John Baker's talk

- Need: better treatment of amplitudes in general

Bibliography

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